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Modelling of the travelling wave piezoelectric motor stator: an integrated review and new perspective

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Articles from different areas which are closely related to the modelling of the stator of travelling wave ultrasonic motors (TWUMs) are reviewed in this work. Thus, important issues relevant to this problem are identified from the areas of vibration of annular plates, laminated plate theories, and modelling of piezoelectric transducers. From this integrated point of view, it becomes clear that there are some very important issues yet to be addressed in the modelling of TWUMs. Firstly, the influence of material properties and stator dimensions on output efficiency, electromechanical coupling coefficients (EMCC) and maximum output energy is to be investigated in more detail. Secondly, the modelling of the electric potential field (by explicitly including the charge equation) for TWUMs seems to be a must for better prediction of displacements and electric fields close to the resonance, as suggested by some recent works [1]. Moreover, the improvement of current models by using shear deformation (or higher order) laminated plate theories (LPTs) in conjunction with approximated methods of solution are discussed.
In addition to analytical models, those works using Finite Element and Finite difference Methods (FEM and FDM) for the modelling and simulation of the TWUM stator dynamics are reviewed.

Keywords: analytical model, piezoelectric motor, resonant actuation

Modelización del estator de un motor piezoeléctrico de onda viajera: una revision integrada y nueva perspectiva

En este trabajo se realiza una revisión de los trabajos de investigación realizados en diversas áreas sobre el modelado del estátor de los motores ultrasónicos de onda viajera (TWUMs). Entre los problemas relevantes que se han estudiado podemos citar la vibración de placas anulares, las teorías de placas laminadas y el modelado de transductores piezoeléctricos. A raíz de este punto de vista integral se hace manifiesto que todavía quedan asuntos importantes que estudiar en el modelado de los TWUMs. En primer lugar, la influencia de las propiedades del material y las dimensiones del estátor en la eficiencia del motor, los coeficientes de acoplamiento electromecánico (EMCC) y la máxima energía entregada deberían ser estudiados más detenidamente. En segundo lugar, el modelado de la distribución del campo eléctrico en los TWUMs (incluyendo la ecuación de carga explícitamente) parece imprescindible para lograr una predicción mejor del desplazamiento y del campo eléctrico cerca de la resonancia, como se ha apuntado en referencias actuales [1]. Además, se discute las mejoras que incorporaría a los modelos existentes en la actualidad la inclusión de las teorías de placas laminadas (LPTs) con deformaciones de corte (o de orden superior), resueltas mediante métodos aproximados. Como complemento a los modelos analíticos, se realiza asimismo una revisión de las técnicas de elementos finitos (FEM) y diferencias finitas (FDM) empleadas en la simulación de la dinámica del estátor de los motores TWUM.

Palabras clave: modelos analíticos, motores piezoeléctricos, excitación en resonancia

1. INTRODUCTION

Travelling Wave Ultrasonic motors (TWUs) belong to a relatively new class of electromechanical devices, which use the inverse piezoelectric effect to obtain linear or rotary motion. Their simple structure, high power density, high torque/low speed operation and its solid-state nature, with no generation of electromagnetic fields, have attracted a wide spread interest.

Behind the simple mechanical construction of TWUMs hides a somewhat elegant working principle [2,3,4]. The rotating motion in these motors is based on the generation of flexural propagating waves on a laminated composite plate (the stator). This is typically made of an elastic layer and one or two piezoelectric layers. The propagating waves are obtained by superposition of two standing waves, which are properly excited in the piezoelectric layer so that they are in quadrature. On the surface of the stator the travelling waves cause an elliptical motion which is then transmitted by direct contact to the rotor. Figure 1 depicts a hollow TWUM of the ring type and shows the typical components of piezoelectric travelling wave motors.

The modelling of piezoelectric motors have been a significant research topic in the last decade. With regard to TWUMs, an important body of knowledge has been achieved on the modelling of the stator dynamics [2,4,5,6,7,8,9,10,11,12], and contact mechanics between stator and rotor [13,14,15,16,17,18,19]. The latter problem is concerned with the frictional transmission of motion and energy at the contact layer, which is in itself a too wide and complicated topic to be reviewed here. Thus, it is left for a future work. We are concerned in this article with the modelling of the stator dynamics.

This work is aimed at reviewing and integrating the most significant articles on modelling of piezoelectric laminated transducers in general and of the stator of TWUMs in particular so that a broader and complete view of this latter problem can be achieved. We believe that such an integrated view is important since, some of the knowledge from closely related areas can be very valuable for the modelling and understanding of common phenomena such as mechanical couplings in laminae, electro-mechanical couplings, shear deformations and rotary inertia effects, electric field variations within laminates, etc..

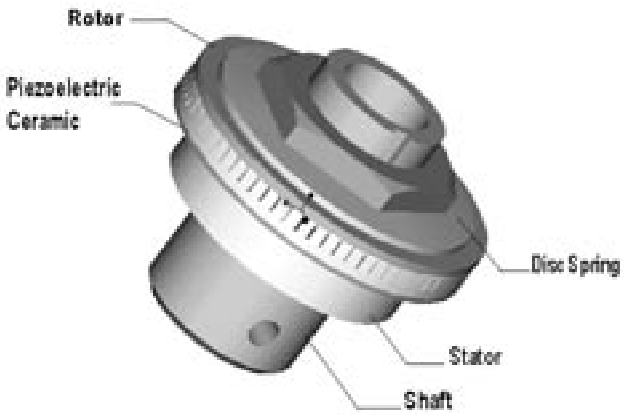

Figure 1: Hollow piezoelectric motor of the ring type showing the typical components of WUMs.

Thus, the interference between the areas of vibration of annular plates, laminated plate theories, modelling of piezoelectric transducers and the modelling of the TWUM stator is examined (see figure 2) . It is hoped that this help to lay out future research directions and expose some of the underlying physical behaviors of piezoelectric motors.

The articles regarding analytical models are reviewed in section 2 and those using the finite element and finite difference method (FEM and FDM) for modelling and simulation of the stator dynamics are presented in section 3. Those works dealing with the modelling of electromechanical laminated transducers are reviewed in subsection 2.1, while those concerning the modelling of heterogeneous laminated plates can be found in subsection 2.2. Special attention is paid to electromechanical couplings and mechanical couplings in subsection 2.1.1 and 2.2.1 respectively. The modelling of the electric potential within piezoelectric layers is reviewed in subsection 2.2.2. In subsection 3.1 the use of the FEM for model simplification is shown, including a practical example. In Section 4 selected articles on the modelling of the stator of TWUMs are reviewed to clearly show the type of models currently found in the literature. Finally, a discussion of main issues and conclusions are presented.

1.1 Modelling of the Stator

The models found in the literature can be first classified as either analytical models or numerical (finite element and finite difference) models. On the one hand, analytical models rely on the use of low order theories and various simplifying assumptions (*e.g.* that the effect of shear deformations and rotary inertia can be neglected), so that closed-form solutions can be obtained. As a consequence, analytical models are usually limited in their capability to model complex geometries, *e.g.*, thick plates, plates with radial varying thickness, asymmetric laminated structure, etc. On the other hand, while finite element analysis (FEA) is imperative for cases of complex geometries, it is inconvenient for motor design. The design of a piezoelectric motor by means of FEA implies that FEA parametric optimization need to be used, which is computationally expensive and time consuming. Thus, the choice of either one of these type of models, depends on the final use of them and in some cases probably a combination of the analytical and numerical approaches could be more appropriated. For instance, the analytical modelling of sub-domains and subsequent numerical integration of the resulting equations is a feasible alternative, which could allow the modelling of ring type motors supported by a thin inner plate.

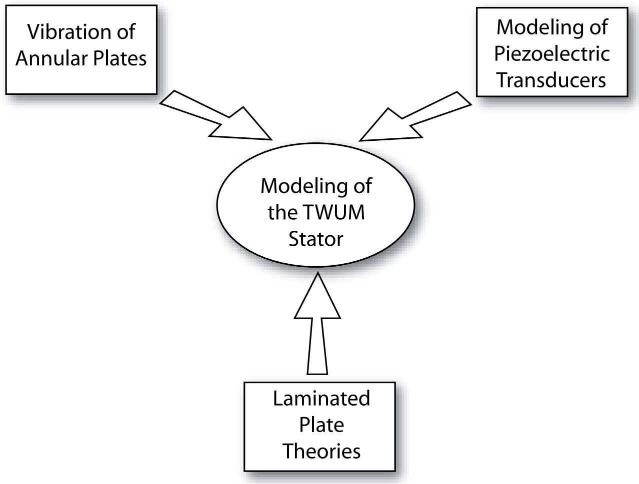

Figure 2: Integrated view of the modelling of TWUM stators.

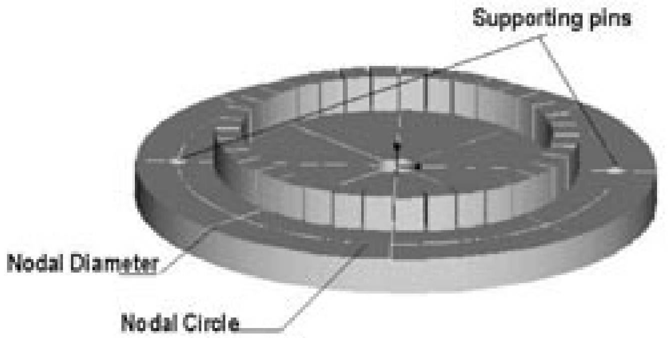

Figure 3: The Panasonic motor is supported by pins at the nodal circle.

2. ANALYTICAL MODELS OF THE STATOR

The stator of a rotary travelling wave ultrasonic motors (shown in figure 1) can be regarded as a composite annular disk which is made of one or two layers of piezoelectric ceramics, typically hard PZT (lead-zirconate-titanate) ceramics, and a layer of an elastic material (*e.g* stainless steel or brass). The disk is generally clamped at its center, which is a rigid node of the structure, by means of a thin support web. However, it could also be supported at a nodal circle (if there is any), as in the Panasonic motor shown in figure 3. In addition, the elastic layer is usually toothed to amplify the tangential velocity of the surface points. The teeth, however, are not explicitly included in analytical models and only their mass is taken into account by lumping it together with the mass of the plate.

2.1. Electromechanical modelling of laminate transducers

The basic electro-elastic composite nature of piezoelectric motors is also shared by other piezoelectric transducers, such as the asymmetric heterogeneous bimorph [20] and the Moonie transducer [3]. Thus, it is not surprising to find an interesting and extensive literature on the modelling of composite piezoelectric-metal transducers of various geometries, specially beams, rectangular plates and disks. Moreover, the analysis and results from some of those works can be valuable to properly model the stator of TWUMs, so they are reviewed in the following paragraphs.

The piezoelectric bimorphs were first studied by C. Baldwin Sawyer in 1931, as quoted by Smits [21] (reference [1] in Smits' article). The first works on the heterogeneous bimorph as a device can be traced back to early 70's. The heterogeneous bimorph, also called unimorph by some authors [22,23] is a beam or plate made of a piezoelectric ceramic layer on top of a nonpiezoelectric elastic layer, as shown in figure 4. W. J. Denkmann *et al* studied the effect of the coupling between extensional and flexural deformations on the dynamic response of a disk shaped transducer made of a two-ply metal/ceramic laminate. They used three analytical methods: uncoupled laminate theory, uncoupled direct variational analysis and the finite element method, and compared the results for different metal-ceramic transducers.

Steel *et al* [24] conducted an experimental and theoretical study of the quasistatic response of the piezoelectric heterogeneous bimorph. They presented results on the bending and stretching of these devices.

Smits *et al* [21] derived the constitutive equations describing the relation between the canonical conjugates (*i.e.* moment and angle of deflection, force and deflection at the tip, volume and uniform distributed force, and charge and voltage at the electrodes) for series and parallel piezoelectric homogeneous bimorphs. Also, in another article co-authored by Choi, Smits [20] presented the constituent equations for heterogeneous bimorphs. In both articles three mechanical boundary conditions are considered: a mechanical moment at the end of the beam, a force applied perpendicularly to the tip of the beam, and a uniform load applied over the length of the beam.

2.1.1. ELECTROMECHANICAL COUPLING AND ENERGY TRANSMISSION COEFFICIENT.

The electromechanical coupling coefficient (EMCC) and the energy transmission coefficient are two very important characteristics of piezoelectric transducers, whose meanings are sometimes confused [3,25]. On the one hand, the square of the EMCC (k^2) is conceptually defined as the ratio of convertible mechanical (electrical) energy stored within the piezoelectric element to the total supplied electrical (mechanical) energy [26]. The EMCC is not a direct measure of the transducer efficiency, but a performance index for the utility of its material and for the transducer bandwidth [25]. Since no mechanical load is applied in this definition, no output work is done. In addition the losses are not included in its definition, so not all the stored energy can be latter used to do work.

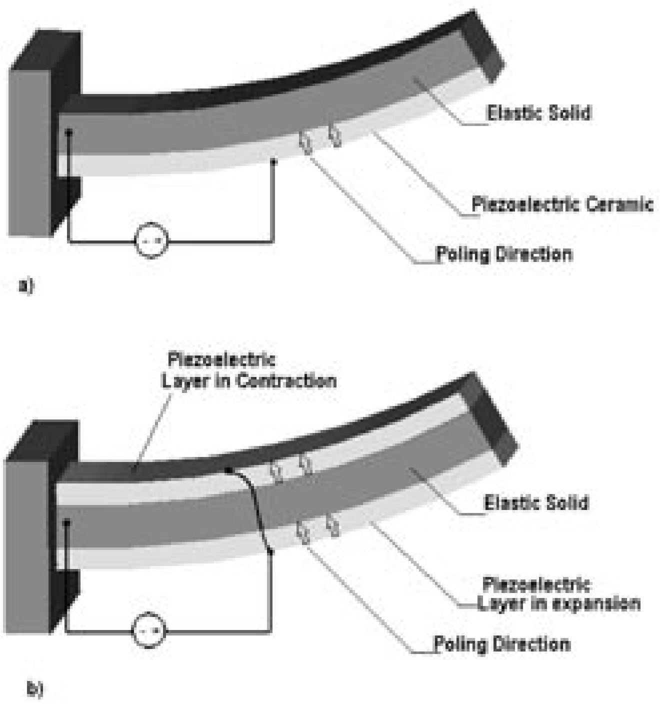

Figure 4: Heterogeneous bimorphs: a) two-layer, b) three-layer.

On the other hand, the energy transmission coefficient (λ_{max}) is defined as the maximum of the ratio of the output mechanical (electrical) energy to the input electrical (mechanical) energy. This is certainly the maximum transducer efficiency.

There are several practical definitions for the EMCC [25,26,27] depending on how the energy ratio (k^2) is calculated. The latest IEEE standard of piezoelectricity (ANSI-IEEE Std. 176, 1987) recommends the use of the Berlincourt *et al* formula [28] for a uniform electroelastic state and the Mason's dynamic formula for close to resonance operation [26]. The former is:

$$k_s = \frac{U_m}{\sqrt{U_e U_d}} \tag{1}$$

where U_m, U_e and U_d are the mutual or interaction energy, the elastic potential energy, and the electric potential energy respectively. Mason's dynamic formula is:

$$k_d^2 = \frac{\omega_a^2 - \omega_r^2}{\omega_a^2} \tag{2}$$

where ω_w is the resonance frequency, and ω_a is the nearest anti-resonance frequency. However, for the analysis of practical transducers a different definition is preferred [25], as it is shown next. The constitutive integrated equations of most transducers can be easily obtained in a four terminal-network expression, of the form:

$$x = C_{11}^{F} + C_{12}E$$
$$q = C_{12}^{F} + C_{22}E$$

for the limit $\omega \to 0$, where x, q, F and E are the mechanical displacement, the electric charge, the mechanical load and the electric field respectively. From these equations, the EMCC is defined as:

$$k_b^2 = C_{12}^2 / C_{11} C_{22} \tag{3}$$

The maximum energy transmission (λ_{max}) and the maximum mechanical output energy (E_{Omax}) can also be obtained from the four-terminal constitutive equations as:

$$\lambda_{max} = 2\left(\sqrt{1/k_b^2} - \sqrt{1/k_b^2 - 1}\right)^2 \tag{4}$$

$$E_{0\max} = \frac{3}{4}\frac{C_{12}^2}{C_{11}}E^2 \tag{5}$$

The computation of the electromechanical coupling and energy transmission coefficients for different electromechanical laminate transducers has been the objective of several investigations. Thus, they are reviewed in the following paragraphs.

Wang *et al* [29] discussed the electromechanical coupling and output efficiency of the bimorph and unimorph actuators based on the constitutive equations developed by Smits [20,21]. Three actuator characteristics are considered, *i.e.* the electromechanical coupling coefficient, the energy transmission coefficient and the mechanical output energy, which are calculated from equations 3, 4 and 5, respectively. It is shown that for the unimorph these characteristics depend not only on the transverse coupling factor k_{31}, but also on the modulus of elasticity ratio of the materials and the thickness ratio of the layers. The results indicate that in order to increase the electromechanical coupling and output mechanical energy of the bender, it is better to chose a stiffer elastic material.

Adelman and Stavsky [30] obtained the axisymmetric motion of a thin disk-shaped unimorph. They used the Kirchhoff assumptions and the Boussinesq's method (reference [6] in Adelman's article) to convert the localized equations of motion and the constitutive equations of the layers to *globalized* plate equations. They plotted the maximum static deflection of the disk as a function of the material mechanical properties and thickness ratio of the piezoelectric and metal layers, for two different types of support. The resulting curves showed only one optimal thickness ratio for each pair of materials. Also the optimal values were strongly dependent on the type of support.

Chang and Chou [31] investigated the dynamic electromechanical characteristics of an asymmetric piezoelectric/elastic laminated beam by electroelasticity and the asymptotic method. The axial expansion-bending coupled motion of the system is separated into quasi-axial expansion and quasi-bending based on the clearly different frequency distribution of the respective vibration modes. They computed the electromechanical coupling coefficient (EMCC) and showed that the capability of energy conversion for a piezoelectric laminate beam decreases as the applied voltage frequency increases. In order to determine the EMCC, they used two different definitions, namely, Mason's dynamic formula 2 and a second method proposed by [27], which defines the EMCC as:

$$k_e^2 = \frac{U^{(d)} - U^{(sh)}}{U^{(d)}} \tag{6}$$

where $U^{(d)}$ is the internal energy of the piezoelectric when the electrodes are disconnected, and $U^{(sh)}$ is the internal energy for short-circuited electrodes. This definition completely agrees with Mason's dynamic formula near resonance and with Berlincourt *et al* formula for the uniform state.

Chang [32] also investigated the dynamic electro-elastic characteristics of the asymmetric rectangular piezoelectric/elastic laminated plate. The theoretical electro-elastic formulation presented in the article uses Kirchhoff-Love hypothesis and assumes the Poisson ratios of the materials are identical to make the coupling coefficients $\mathbf{B}_{ij}$ vanish. Moreover, they neglect the coupling inertia since the corresponding term $\mathbf{R}_1$ is considered much smaller than the translational inertia $\mathbf{R}_0$. The eigenvalues and eigenfunctions were determined by means of the extended Kantorovic method.

2.2. Vibration of heterogeneous laminated plates

In order to study the vibration of thick heterogeneous plates (such as those found in TWUM's stators), sufficiently accurate theories and models need to be chosen, so that the effect of shear deformation, rotary inertia, and the different couplings between deformations are included.

In general, shear deformation and rotary inertia may be neglected for stators where the thickness is small compared to the outer diameter and to the wavelength of the highest mode of interest [33,34]. As shown by Mindlin [34] when the ratio of the wavelength to the thickness of the plate is less than 5 the effect of shear deformation and rotary inertia becomes important. Thus, the classical plate theory would overestimate natural frequencies and underestimate deformations when used for such a case [33,35].

The typical asymmetrical layering of the stator generally results in a coupling between extensional and flexural deformation, as well as coupling between extensional and rotary inertia [36]. One of the effects of the deformational coupling is the reduction in the effective flexural stiffness, which in turn lowers the natural frequencies of the plate.

Several fine theories can be used to model asymmetric laminated plates: the classical laminated plate theory (CLPT), which is attributed in its complete form to Reissner and Stavsky [37]; first order [38] and higher order shear deformation theories [33,39,40], with and without rotary inertia, are probably the most important approaches. However, it is too difficult to obtain closed-form solutions with these theories. Thus, few authors have modeled the stator of piezoelectric motors as a laminated structure and they used only an approximation of the simplest laminated theory (*i.e.*, CLPT)[6].

2.2.1. MECHANICAL COUPLING IN LAMINATED PLATES

In order to properly show the different mechanical coupling terms which could be present in a laminated plate, the formulation and notation of a first order shear deformation theory in cylindrical coordinates is now introduced.

The coordinates and layer numbering conventions are shown in figure 5. The rθ plane coincides with the midplane of the plate. The displacement field is specified as,

$$\begin{aligned} u_1(r,\theta,z,t) &= u_r(r,\theta,t) + z\psi_r(r,\theta,t) \\ u_2(r,\theta,z,t) &= u_\theta(r,\theta,t) + z\psi_\theta(r,\theta,t) \\ u_3(r,\theta,z,t) &= u_z(r,\theta,t) \end{aligned}$$

where, u_1, u_2 and u_3 are the displacement in the r, θ, and z direction respectively; u_r, u_θ and u_z are the corresponding midplane displacement; and ψ, and ψ_θ are the rotations in the rz and θz planes respectively. The strain-displacement relations are then derived as:

$$\begin{aligned} \varepsilon_1 &= \frac{\partial u_r}{\partial r} + z\frac{\partial \psi_r}{\partial r} = \varepsilon_1^0 + z\kappa_1^0 \\ \varepsilon_2 &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) + \frac{z}{r}\left(\frac{\partial \psi_\theta}{\partial \theta}\psi_r\right) = \varepsilon_2^0 + z\kappa_2^0 \\ \varepsilon_3 &= \frac{\partial u_z}{\partial z} \\ \varepsilon_4 &= \psi_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \varepsilon_5 &= \psi_r + \frac{\partial u_z}{\partial r} \\ \varepsilon_6 &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + z\left(\frac{1}{r}\frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right) = \varepsilon_6^0 + z\kappa_6^0 \end{aligned} \tag{7}$$

where the six strain components ε_i (i=1,2,..,6) are ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz}, $2\varepsilon_{\theta z}$, $2\varepsilon_{rz}$ and $2\varepsilon_{r\theta}$ respectively; and ε_i^0 and κ_i^0 are the midplane strains and curvatures respectively.

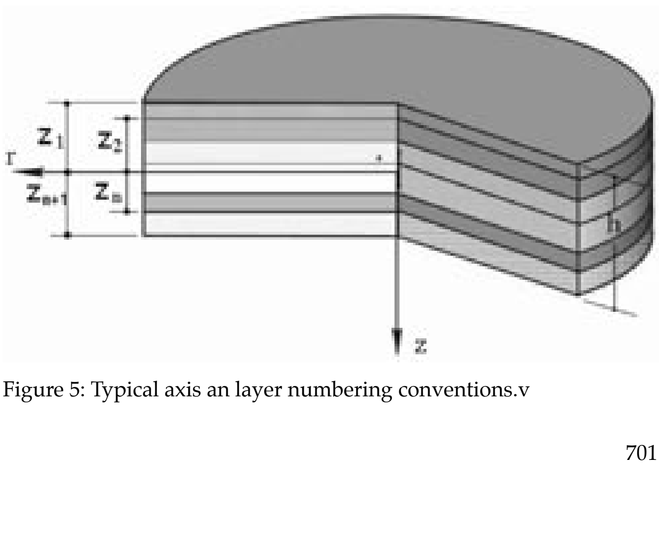

Figure 5: Typical axis an layer numbering conventions.v

The equations of motion are then obtained by means of the Hamilton's principle:

$$\int_{t_0}^{t_1} \delta L dt = 0 \tag{8}$$

where the first variation of the Lagragian δL is found as:

$$\begin{aligned}\delta L &= \delta\Pi - \delta K \\ &= \int_R \left(\delta\varepsilon_1^0 N_1 + \delta\kappa_1^0 M_1 + \delta\varepsilon_2^0 N_2 + \delta\kappa_2^0 M_2 + \delta\varepsilon_4^0 Q_2 + \delta\varepsilon_5^0 Q_1 + \delta\varepsilon_6^0 N_6 + \delta\kappa_6^0 M_6 - \delta\omega q\right) dxdy \\ &- \int_{C_1} \hat{N}_n \delta u_n dS - \int_{C_2} \hat{N}_s \delta u_s dS - \int_{C_3} \hat{M}_n \delta\psi_n dS - \int_{C_4} \hat{M}_s \delta\psi_n dS \\ &- \int_{C_5} \hat{Q}_n \delta\omega dS - \rho \int_V \left[u_1 \delta u_1 + u_2 \delta u_2 + u_3 \delta u_3\right] dV\end{aligned} \tag{9}$$

here Π and K are the potential and kinetic energies; N_i, Q_i and M_i are the force and moment resultants; q is the distributed transverse force; and the integral terms of regions from C_1 to C_5 corresponds to the natural boundary conditions (i.e. moments and force resultants). The equations of motion and boundary conditions can then be obtained by substituting equation 7 into equation 9, then substituting this result into 8, and integrating by parts.

The constitutive equations of a laminated plate (see for instance, [33,37,41]) can be written in matrix form as:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B]^T & [D] \end{bmatrix} \begin{Bmatrix} \{e\} \\ \{\kappa\} \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} A_{55} & A_{45} \\ A_{45} & A_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_5 \\ \varepsilon_4 \end{Bmatrix} \tag{11}$$

The matrix **[A]**, **[B]** and **[D]** can be calculated from the generalized plane stress moduli c_{ij} of the n layers as:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{m=1}^{n} \int_{z_m}^{z_{m+1}} \left(1, z, z^2\right) c_{ij}^{(m)} dz \qquad (i, j = 1,2,6) \tag{12}$$

$$A_{ij} = \sum_{m=1}^{n} K_i K_j \int_{z_m}^{z_{m+1}} c_{ij}^{(m)} dz \qquad (i, j = 4,5) \tag{13}$$

where $\mathbf{A}_{ij}$ (i,j=4,5) denote the stiffness coefficients and $\mathbf{K}_i$ are the shear correction factors.

It can be observed that, in order to eliminate the coupling between extensional and flexural deformation the matrix **[B]** must be zero. It turns out that, there are two cases for which matrix **[B]** becomes zero, firstly if all materials making up the laminate plate have the same Poisson ratio ν, there exists a reference neutral plane which completely uncouples the deformation. Secondly, if the laminates are symmetric (in thickness and material) with respect to the midplane, the deformations also become uncoupled [23].

On the other hand, the translational, coupling and rotary inertia $\mathbf{I}_0$, $\mathbf{I}_1$ and $\mathbf{I}_2$, respectively, can be expressed as:

$$\left(I_0, I_1, I_2\right) = \sum_{m=1}^{n} \int_{z_m}^{z_{m+1}} \left(1, z, z^2\right) \rho dz \tag{14}$$

In this case, even if the Poisson ratios of the materials are different the coupling inertia $\mathbf{I}_1$ becomes null when the reference plane is chosen at:

$$h_0 = \frac{\sum_{m=1}^{n} \int_{z_m}^{z_{m+1}} \rho dz}{\sum_{m=1}^{n} \int_{z_m}^{z_{m+1}} \rho z dz} \tag{15}$$

2.2.2. STRAIN INDUCED METHODS AND MODELLING OF THE ELECTRIC POTENTIAL

In order to model heterogeneous laminated plates, which are made of piezoelectric and nonpiezoelectric layers, the piezoelectric effect needs to be considered in the constitutive relationships. For a piezoelectric layer, the constitutive equations can be found from the electromechanical coupling, dielectric and mechanical stiffness matrices as:

$$\begin{Bmatrix} D_3 \\ \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} k_3^S & e_{31} & e_{31} & \\ -e_{31} & C_{11}^E & C_{12}^E & \\ -e_{31} & C_{21}^E & C_{22}^E & \\ & & & C_{66}^E \end{bmatrix} \begin{Bmatrix} E_3 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \tag{16}$$

where c_{ij}, e_{ij} and k_i are the elastic, piezoelectric and dielectric permittivity constants, respectively. In these equations the plane stress values for the constants need to be used [5] and the superscripts E and S refer to values taken at constant electric field and constant strain, respectively. In addition, the Lagragian of equation 9 needs to be augmented with the terms corresponding to the electric energy stored within the piezoelectric ceramic and the potential energy due to electromechanical coupling. These terms can be found to be:

$$W_E = \frac{1}{2} \int_{V_p} E^T D dV_p \tag{17}$$

$$\Pi_p = -\frac{1}{2} \int_{V_p} S^T e^T E dV_p \tag{18}$$

where **S** is the strain matrix. The piezoelectric constants **e** are determined from: $e = dc^E$, where $\mathbf{d} = [d_{31}\ d_{31}\ 0]$ for most PZT ceramics.

Most of the models for heterogeneous laminated plates use strain induced methods, i.e. they include the piezoelectric effect in the constitutive relationships only and the charge equations of electro-elasticity are usually ignored, as in the formulation just presented.

It is well known that, for most piezoelectric structures, the elastic wavelengths are physically much smaller than the electromagnetic wavelengths. Consequently, the quasi-static approximation of Maxwell's equations can be used [42]. Thus, the electric field and charge equations become:

$$E = -\nabla\phi, \ \nabla D = 0$$

where **E**, ϕ and **D** are the electric field, the scalar potential function and the electric displacement, respectively.

The decision of whether to model the electric potential field or not mainly depends on the thickness of the piezoelectric layer. A typical piezoelectric material used for TWUM's (i.e. PZT ceramics) is relatively stiff, when compared against PVDF (polyvinyledene flouride) for instance, so the thicker the layer the smaller the effective electric field [39], which in turn reduces the resulting deflections. Moreover, the coupling of the charge equations and the momentum equations is increased through the dielectric constant, for that case. Some of the works which include the modelling of the electric field are mentioned in the following paragraphs.

Tiersten (1969) [43] modeled single-layer piezoelectric plates and included the charge equation. Also, Tzou and Zhong (1993) [44] used first-order shear deformation to model piezoelectric shells and included the charge equation. The plate equations were then derived for single piezoelectric layer. Most recently, J. A. Mitchell and J.N. Reddy (1995) [39] presented a refined third-order laminated plate theory, including electric potential. The theory accounts for shear deformation without using correction factors and model the potential function on a discrete layer approximation, equivalent to modelling the variation of this function with 1-D finite elements.

M. C. Ray *et al* presented a static analysis of a simply supported "smart" infinite plate (i.e. a substrate of elastic material sandwiched between two layers of piezoelectric materials), under cylidrincal bending. The material of sensor and actuator layers is PVDF. The electric potential functions across the thickness of the sensor and actuator layer were found to be fairly linear.

P. Heyliger and D.A. Saravanos (1995)[45] developed exact solutions for predicting the coupled electromechanical vibration characteristics of simply supported rectangular laminated plates with embedded piezoelectric layers. They used the constitutive equations for piezoelectric materials and the charge equation of electrostatics, assumed for each layer, to solve the 3-D problem of elasticity. At the interfaces between layers, the continuity conditions on displacement, traction, potential, and electric displacement were established. The modal analysis is solved by assuming that the displacement components and the electrostatic potential have the following form:

$$u(x,y,z,t) = Ue^{j\omega t}e^{sz}\cos(px)\sin(qy)$$
$$v(x,y,z,t) = Ve^{j\omega t}e^{sz}\cos(px)\sin(qy)$$
$$w(x,y,z,t) = We^{j\omega t}e^{sz}\cos(px)\sin(qy)$$
$$\phi(x,y,z,t) = \Phi e^{j\omega t}e^{sz}\cos(px)\sin(qy)$$

where U, V, W and Φ are constants, s is an unknown to be determined, $p=m_x\pi/l_x$, $q=m_y\pi/l_y$, m_x and m_y are the positive integers used for the modes of vibration. The substitution of these expressions into the structure differential equations yields a system of equations whose nontrivial solution (when the determinant is zero) is then used to determine the value s and the natural frequencies based on an iterative scheme. The results of this study also demonstrated the need for a proper modelling of the electric potential within the piezoelectric layers.

J. H. Huang [46] and T. Wu studied the fully coupled response characteristics of composites plates made of fiber response composite laminates and piezoelectric layers. They assumed that the electric potential has a quadratic distribution through the thickness direction, which is based on the work of Rogacheva [47], who observed a linear distribution of the electric field.

The commonly used assumption of linear variation of potential inside piezoelectric layers was also questioned by S. V. Gopinathan *et al* (1999) [1]. The mathematical formulation used to obtain the governing equations in their work is similar to that of Heyliger's article [45] but they include a damping term in the dynamic field equations. The stress, strain and electric field distributions of a simply supported rectangular single layer of piezoelectric material and a three layered heterogeneous laminate are evaluated under harmonic forcing potentials. In the latter case, the laminate consists of a composite orthotropic layer sandwiched between two piezoelectric layers. In addition, two different forcing frequencies were considered, one much lower ($0.5\omega_1$) than the first bending mode frequency and the other closer ($0.95\omega_1$) to that frequency. They observed that close to the resonant frequency the electric field is not constant inside the piezoelectric layer.

In a most recent paper Gopinathan *et al* (2000) [1] reviewed different laminated plate theories used for the modelling of laminated composite beams and compared the results of using a first order shear deformation theory (FSDT) with the above mentioned 3D formulation, which explicitly includes the charge equation. They found that for a 3-layer very thin beam, with an aspect ratio ($AR=l/h$) of 50 the FSDT gives the same electric field values than the 3D formulation and estimates the first four bending mode frequencies with an error of less than 3%. They also reported that when a moderately thin plate ($AR=20$) was excited at 90% of the first resonance frequency, the FSDT predicts the inplane displacements, transverse displacements and electric potential with errors of 45%, 25% and 42% respectively, which are quite large and should be experimentally confirmed.

3. FINITE ELEMENT AND FINITE DIFFERENCE ANALYSIS OF THE STATOR

As mentioned earlier the stator can also be modelled by means of the finite element method (FEM). The stress, strain, and electric field within the structure are readily obtained in the post-processing stage of program execution. Of course any given shape and type of material can be selected for design. The main drawback of this method is that parametric optimization of the structure is computationally expensive and time consuming. For instance, during mechanical design of the motor, it is desirable to know, for a given set of specifications, the thickness and young module ratios of the layer materials, the inner radio and the axial external load that yield a maximum energy conversion efficiency. Thus, for each iteration of the search engine a new mesh needs to be generated, its resonant frequency needs to be found and the displacement of the nodes, element solution and electric fields needs to be found for the resonant frequency. The FEM is very useful though for analysis of a given structure and prediction of the final displacement of the stator and the rotor as well.

J. Krome and J. Wallaschek [12] used the 3-D finite element method to investigate the influence of the shape of the piezoelectric ceramic on the vibration of the stator. By means of the variational principle, they first found the coupled finite element matrix equation which includes the mechanical and electrical degrees of freedom. Afterward, through static condensation of the electrical degrees of freedom, the equations were reduced to a generalized eigenvalue problem.

They compared the eigenmodes and amplitudes of the harmonic response for two cases, a continuous ceramic ring and independent ceramic segments. In the latter case the eigenmode is not symmetric (there appeared symmetry disturbances) and the amplitude of the flexural vibration is smaller than for the continuous ceramic. Symmetry of the eigenmode is very desirable to yield an uniform contact between stator and rotor.

While some authors prefer to use 3-D FEM to accurately predict the modal frequencies and transient response of the stator [12,49], others have tried to avoid the drawbacks above mentioned, by using modified finite elements and finite difference techniques. Y. Bar-Cohen *et al* (1998) [50] used modified annular finite elements which are based on the symmetrical characteristics of rotary piezoelectric motors. The details of the method can be found in [51]. Hagedorn *et al* [8] approximated the partial differential equation of the varying thickness plate by centered finite differences. They split the radius into intervals so that the thickness $h(r)$ is piecewise continuous. The rotary inertia was included in the formulation but the shear deformations were neglected.

In contrast to 3-D FEM, were frequencies are found in increasing order, the last two alternative methods allow to choose a specific vibrational mode prior to the calculation of its frequency and mode shape.

3.1. Use of FEA to evaluate model simplification

Apart from using FEA for prediction of the response of a particular stator design, the finite element method can also be used for sensitivity analysis. Thus, if a particular element of the stator (*e.g.* teeth, chamfers, supporting web, step etc.) does not greatly influence the harmonic response or the natural frequencies of the structure, it could be omitted or the structure could be somehow simplified for that particular analysis.

A practical example is presented now of the use of ANSYS56 FEM program to evaluate a model simplification, which consists on the removal of the supporting inner plate (shown in figure 6) when determining the modal frequencies of the stator.

The main hypothesis is that, between a specific range of relative dimensions of the inner web, the flexural mode frequencies of a clamped stator could be found by omitting the inner web and assuming that the inner and outer surfaces, *r=a* and *r=b*, are free of constraints. In other words, the question is: are the flexural mode frequencies of a "weakly" inner supported stator close to those of an "unsupported" stator?

The influence of the thickness of the inner plate on the mode frequency was first examined. Three different values of the thickness ratio t_s/t_w were used while others dimensions were kept fixed at *a*=19.5, *b*=25 and *c*=35 mm. The results from the modal analysis for these values is presented in Table I. The values in parenthesis are the percent of error of each frequency with respect to the corresponding flexural mode frequency of the unconstrained annular plate. Its is remarkable that for a relatively small thickness ratio (t_s/t_w =6) the error is less than 5%.

TABLE I: FLEXURAL MODE FREQUENCIES FOR DIFFERENT t_s/t_w RATIOS (*a*=19.5).

b=25 mm, c=35 mm	**Flexural Mode Freq.** (kHz)		
	n=3	n=4	n=5
Free-free	21.40	39.50	60.79
Clamped, t_s/t_w = 10	21.63(1.1)	39.61(.30)	60.92(.21)
Clamped, t_s/t_w = 7.5	21.91(2.4)	39.60(.25)	61.39(.98)
Clamped, t_s/t_w = 6.0	22.27(4.1)	39.65(.38)	60.82(0.05)

There are, of course, other dimensions that could further increase the frequency difference between the inner clamped and the free annular plate. Thus, the influence of the plate diameters *a*, *b* and *c* over the flexural mode frequencies was considered by comparing the results for different values of the ratio *c/(b-a)*. The results of this last investigation are presented in Table II.

TABLE II: FLEXURAL MODE FREQUENCIES FOR DIFFERENT *c/(b-a)* VALUES.

b=25 mm, c=35 mm	**Flexural Mode Freq.** (kHz)		
	n=3	n=4	n=5
Free-free	21.40	39.50	60.79
Clamped, *c/(b-a)* = 6.36	21.63(1.1)	39.61(.30)	60.92(.21)
Clamped, *c/(b-a)* = 1.7	21.08(1.5)	39.55(.12)	61.05(.42)
b=17.5 mm, c=35 mm			
Free-free	25.28	45.65	69.99
Clamped, *c/(b-a)* = 2.6	25.09(.63)	44.86(1.7)	68.56(2.0)

It can be observed that the resulting percent of error, for a relatively wide range of dimensions and vibration modes remains less than 5%, and for very thin inner plates t_s/t_w = 10 is less than 1%.

4. DISCUSSION

In this section, some important issues which have emerged from the review are discussed as an attempt to outline future directions for improvement of the modelling and design of TWUMS.

• Even though, to the best of our knowledge, only one author [6] used laminated plates theories (LPTs) to model TWUMs, we believe that by using this approach an important insight is gained on the role of the different variables of the stator. Thus, it is easy to see for instance, how the material properties and dimensions of the laminates affect the mechanical couplings of the plate. For non-symmetric laminated plates, in general, the extensional and flexural deformations are coupled through matrix **[B]** in the constitutive equations 10. Thus, the extensional strains induced by the piezoelectric layer lead to moment resultants **{M}** in the composite plate. It can be seen that these moments could either increase or decrease the bending deformations depending on the sign of the elements of matrix **[B]**. These elements, in turn, depend on the elasticity modulus and thickness ratio, and Possion coefficients of the materials making of the laminate. In addition, by using LPTs it is straightforward to add laminates, to take into account adhesive layers for example. In such a case only the elements of **[A]**, **[B]** and **[D]** need to be modified.

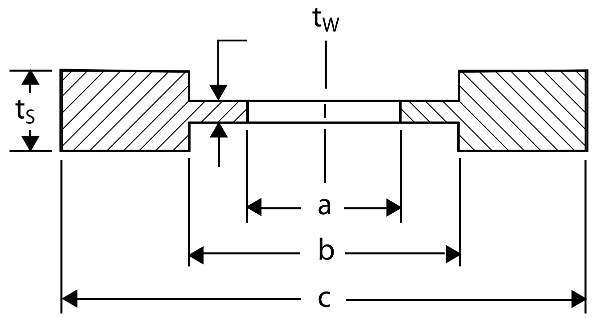

Figure 6: Cross section showing the inner supporting plate

• The relation of mechanical parameters (i. e. geometry and material properties) to the efficiency, electromechanical coupling and output power of the stator is another important related area which deserves more attention. The few contributions are mainly on the optimization of an estimated output efficiency of the motor as function of the thickness ratio only [9,10,11]. However, the influence of other dimensions and material properties (*e.g.* outer diameter to inner diameter ratio and modulus of elasticity ratio) on the efficiency and other performance measures needs to be investigated. In that sense, since a ring type TWUM can be thought as a "circular" beam with joined ends, the work of Wang *et al* [29] for heterogeneous bimorphs can be somehow extended to TWUMS. They found that in order to increase the electromechanical coupling and output mechanical energy of the bender, it is better to choose a stiffer elastic material.

• In order to accurately predict the resonant frequencies and modal displacements, it is important to include the effects of shear deformation and rotary inertia for thick stators. When shear deformation and rotary inertia are not taken into account, the model underestimate deformations and overestimate the resonant frequencies [33,35]. In fact for a particular plate the shear deformation plate theories produce more modal frequencies (i.e. more vibration modes) on a given frequency range [35]. Analytical closed form solutions for shear deformation and higher order laminated plate theories (LPTs) are too difficult to obtain. However, approximated solutions based on methods such as the Ritz`s method and the finite difference method are good alternatives. These approaches allow not only to include the effects of shear deformation and rotary inertia, but also to model radial variations of the plate thickness. In addition, since this approximated methods are computationally less expensive than the FEM, parametric optimization is possible.

• For thick PZT ceramics, when the electric field is not properly modelled the resulting deflections are overestimated and the couplings in the charge and momentum equations are underestimated. Thus, for thick piezoelectric layers the assumption of a constant electric field is not adequate, specially close to the resonance. The results reported by Gopinathan *et al* [1] suggest that the theories currently used to model TWUM stators, whose Ars are typically less than 20, give rather poor prediction of the displacements and electric potentials for excitation frequencies close to the resonant frequency.

5. CONCLUSIONS

Articles from different areas which are closely related to the modelling of the stator of TWUMs were reviewed in this work. Thus, important issues relevant to this latter problem were identified from the areas of vibration of annular plates, laminated plate theories, and modelling of piezoelectric transducers.

Some possible future direction for improvement of current models and general needs are enumerated now.

i. The effect of different material properties and dimensions of the laminates on the efficiency, electromechanical coupling, and output power of the stator is to be investigated in more detail.

ii. The charge equation needs to be explicitly included in the mathematical formulation to properly predict the displacement and the electric field inside the piezoelectric layers, close to resonance, as suggested by recent works [45,1].

iii. The use of shear deformation (or higher order) LPT together with approximated solution methods such as Ritz's and finite difference methods would allow to model radial variation of the stator thickness and perform parametric optimization of the stator.

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